Exercise 2

Solve $(1 + t)u_t + xu_x = 0$. Then solve it with the initial condition $u(x, 0) = x^5$ for $t > 0$.

Solution

For a function of two variables $u = u(x, t)$, its differential is defined to be

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial t}\,dt.$$

Divide both sides by dt.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

This equation gives the relationship between the total derivative of u with respect to t and its partial derivatives. Divide both sides of the given PDE by $1 + t$.

$$u_t + \frac{x}{1 + t}u_x = 0$$

Comparing this PDE with the right side of the previous equation, we see that on the curves (characteristics) in the xt-plane that satisfy

$$\frac{dx}{dt} = \frac{x}{1 + t}, \tag{1}$$

the PDE simplifies to an ODE.

$$\frac{du}{dt} = 0, \tag{2}$$

u can be solved for by integrating both sides of equation (2) with respect to t.

$$u = f(\xi),$$

where f is an arbitrary function to be determined and ξ is a characteristic coordinate. The characteristic curves are found by solving equation (1). Use the method of separation of variables.

$$\frac{dx}{x} = \frac{dt}{1 + t}$$

Integrate both sides.

$$\ln|x| = \ln(1 + t) + C$$

Exponentiate both sides.

$$|x| = e^{\ln(1+t)+C}$$

Eliminate the absolute value sign by introducing $\pm$ on the right side.

$$x = \pm e^C e^{\ln(1+t)}$$

Use a new constant of integration ξ. Thus, the characteristic curves are

$$x = \xi(1 + t).$$

Solve this equation for ξ.

$$\xi = \frac{x}{1 + t}$$

The general solution to the PDE is then

$$u(x,t) = f\left(\frac{x}{1+t}\right).$$

The aim now is to determine the particular function f that satisfies the initial condition.

$$u(x,0) = f(x) = x^5$$

Though this is in terms of x, the equation is actually $f(w) = w^5$, where w is any expression we choose:

$$f\left(\frac{x}{1+t}\right) = \left(\frac{x}{1+t}\right)^5.$$

Therefore,

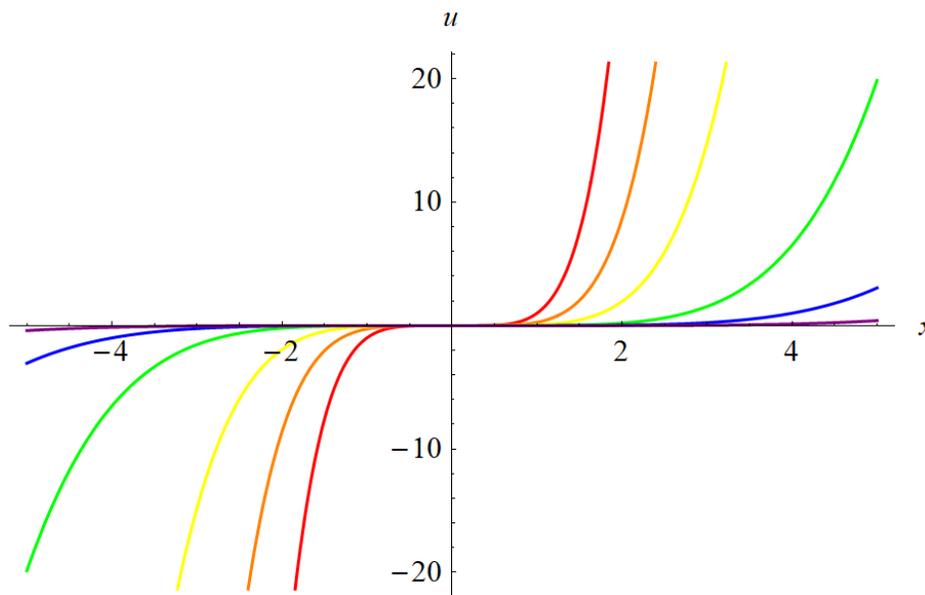$$\boxed{u(x,t) = \left(\frac{x}{1+t}\right)^5.}$$

Figure 1: This is a plot of the solution u as a function of x for various times. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.3$, $t = 0.75$, $t = 1.75$, $t = 3$, $t = 5$, respectively.

The solution to the PDE starts as a quintic polynomial $u = x^5$ and decays to $u = 0$ as t goes to infinity.
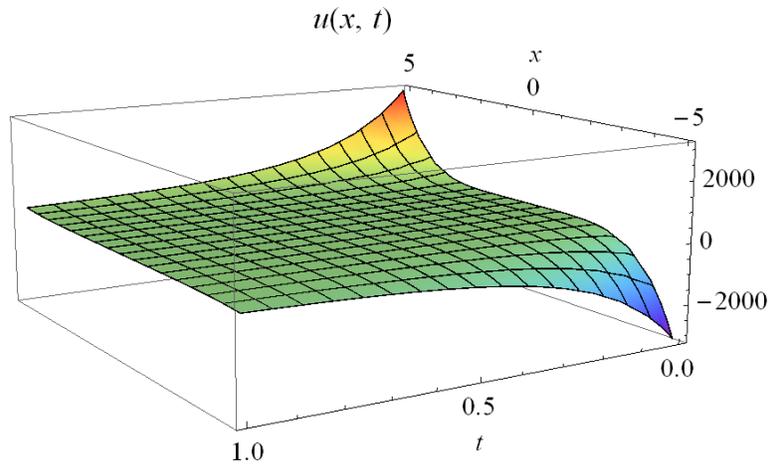
Figure 2: This is a plot of the two-dimensional solution surface $u(x, t)$ in three-dimensional space for $-5 < x < 5$ and $0 < t < 1$.

To determine where in the xt-plane the solution to the PDE is valid, it's necessary to plot the characteristic curves along with the given data curve at $t = 0$.

$$x = \xi(1 + t)$$

The way to plot them is to choose a certain value for ξ and then to plot the resulting equation in the xt-plane. This is done over and over until the plane is full. A computer can accomplish this task very efficiently. ξ is chosen here to go from -10 to 10, incrementing by 0.25 each time.
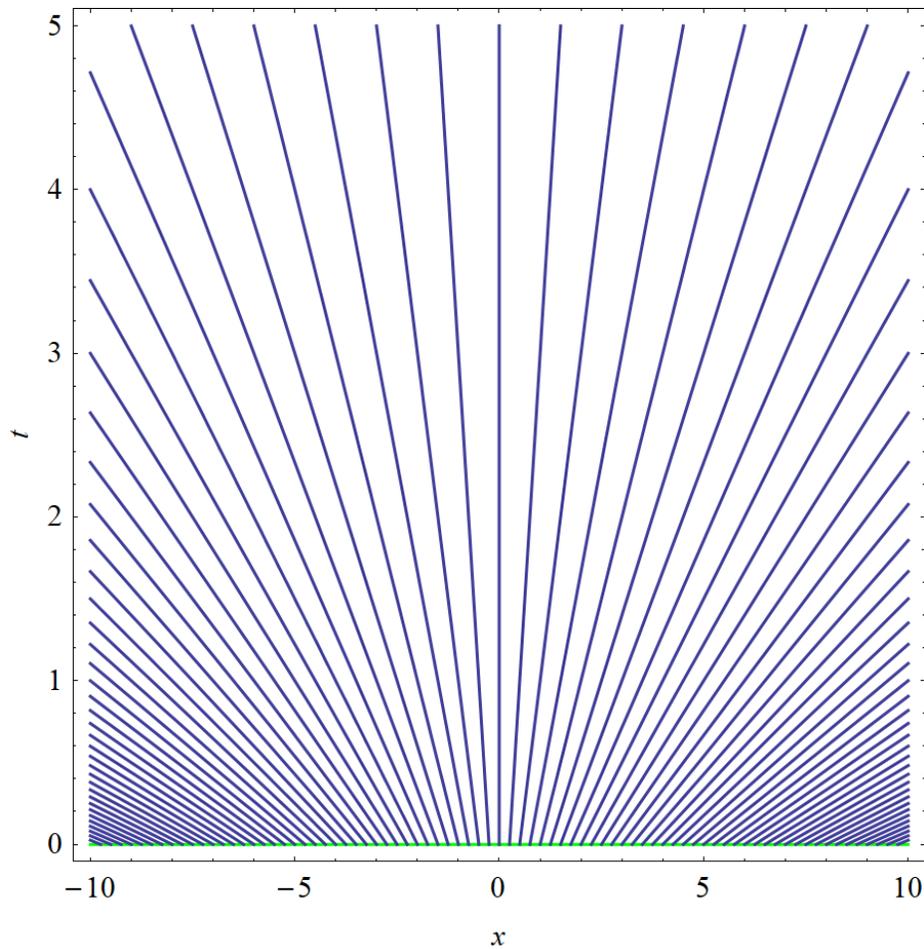
Figure 3: Plot of the characteristic curves in the xt-plane along with the given data curve $(t = 0)$ in green.

There are a few things to observe here. All the characteristics are straight lines. This is because the right side of the PDE is 0. The solution to the PDE is unique and constant along each of the characteristics. Because each of the characteristics intersects the data curve exactly once and extends throughout the upper half-plane, the boxed solution we have for the PDE is valid for all x and $t > 0$.